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Complete cosmic history with a dynamical $\Lambda = \Lambda(H)$ term

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In the present mainstream cosmology, matter and space-time emerged from a singularity and evolved through four distinct periods: early inflation, radiation, dark matter, and late-time inflation (driven by dark energy). During the radiation and dark matter dominated stages, the universe is decelerating while the early and late-time inflations are accelerating stages. A possible connection between the accelerating periods remains unknown, and, even more intriguing, the best dark energy candidate powering the present accelerating stage (Λ-vacuum) is plagued with the cosmological constant and coincidence puzzles. Here we propose an alternative solution for such problems based on a large class of time-dependent vacuum energy density models in the form of power series of the Hubble rate, $\Lambda = \Lambda(H)$. The proposed class of $\Lambda(H)$-decaying vacuum model provides: (i) a new mechanism for inflation (different from the usual inflaton models), (ii) a natural mechanism for a graceful exit, which is universal for the whole class of models; (iii) the currently accelerated expansion of the universe, (iv) a mild dynamical dark energy at present; and (v) a final de Sitter stage. Remarkably, the late-time cosmic expansion history of our class of models is very close to the concordance ΛCDM model, but above all it furnishes the necessary smooth link between the initial and final de Sitter stages through the radiation- and matter-dominated epochs.

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I. INTRODUCTION

Several cosmological observations (supernovae type Ia, CMB, galaxy clustering, etc.) have converged to a paradigm of a cosmic expansion history that involves a spatially flat geometry and a recently initiated accelerated expansion of the universe [1–11]. This expansion has been attributed to an energy component called dark energy (DE) with negative pressure, which dominates the universe at late times. The easiest way to fit the current cosmological data is to include in the Friedmann equations the cosmological constant (CC) [9–11]. Despite the fact that the so-called concordance model (or ΛCDM model) describes well the global properties of the observed universe, it suffers from the CC problem [12,13]. However, the alternative frameworks (e.g., quintessence models and the like) are not free from similar fine-tuning and other no less severe problems (including the presence of extremely tiny masses). Whichever way it is formulated, the CC problem appears as a tough issue which involves many faces: not only the problem of understanding the tiny current value of the vacuum energy density ($\rho_\Lambda = c^2\Lambda/8\pi G \simeq 10^{-47}$ GeV^4) [13] in the context of quantum field theory (QFT) or string theory, but also the cosmic coincidence problem, i.e., why the density of matter is now so close to the vacuum density [14].

Even before the discovery of the accelerating universe based on Supernovae observations (see [2–5] and Refs. therein), a great deal of attention was dedicated to time-evolving vacuum models, $\Lambda \equiv \Lambda(t)$, motivated basically by the age of the universe and CC problems [15–25] (see also [26] for a short review of this earlier literature). These models also act as an important alternative to the cosmic concordance (ΛCDM) and scalar-fields dark energy models, since they can explain in an efficient way the accelerated expansion of the universe and also provide an interesting attempt to evade the coincidence and cosmological constant problems of the standard Λ-cosmology (see, for instance, Lima in [1]).

Although the precise functional form of $\Lambda(t)$ is not known, which is however also the case for the vast majority

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of the usual dark energy models, an interesting QFT approach within the context of the renormalization group (RG) was proposed a long time ago [27,28]. Later on, the RG-running framework was further explored in [29–34] from the viewpoint of QFT in curved space-time by employing the standard perturbative RG-techniques of particle physics (see [35,36] for recent reviews). These RG-based dynamical vacuum energy models emphasize on the evolution of the vacuum energy as a particularly well-motivated function of the Hubble rate, i.e., $\Lambda(t) = \Lambda(H(t))$, namely functions containing even powers of H and including also an additive constant term. These proposals were confronted with the first supernovae data in [32], and later on with the modern observations on supernovae, baryonic acoustic oscillations, CMB, and structure formation in [37–41]. Variants of these models facing efficiently the cosmic coincidence problem and some aspects of the CC problem also exist in the literature [42,43], including the implications on the possible variability of the fundamental constants [44]. As remarked before, there is an extensive (old and new) literature in which the time-evolving vacuum has been phenomenologically modeled as a function of time in various possible ways, in particular, as a function of the Hubble parameter [18–26,45–56].

Technically speaking, it would be important if we could find a way to unify all the stages of the history of the universe within the generic framework of the running vacuum models, as these are the closest ones to fundamental QFT physics. While a first formulation of this unification was given in [55,56], the aim of the current work is to put forward a large class of models of this kind in which the vacuum dynamics of the early universe is linked with that of the late universe in a way fully consistent with the phenomenological observations. At the same time we suggest possible clues to solve or alleviate some of the fundamental problems of the early universe, most particularly the transition from the inflationary epoch to the standard radiation epoch. It starts from a nonsingular inflationary stage which has a natural (universal) ending into the radiation phase (thereby alleviating the horizon and graceful exit problems), and, finally, the small current value of the vacuum energy density can be conceived as a result of the massive disintegration of the vacuum into matter during the primordial stages.

The plan of the paper is as follows. In Sec. II we discuss the energy conservation in general dynamical models of the vacuum energy, whereas in Secs. III and IV we motivate in different ways the form $\Lambda = \Lambda(H)$ we are interested in. In Secs. V to VII we provide the analytical solutions in the early and late universe respectively (the formulation in terms of an effective potential is presented in Sec. VI). The summary and general discussion is provided in Sec. VIII. Finally, in the appendix we furnish some additional technical details related to the derivation of the cosmological equations for the models under consideration.

II. MODELS WITH DYNAMICAL VACUUM ENERGY

In the current article we would like to investigate the cosmic expansion within the context of the time varying vacuum energy density. To start with, let us model the expanding universe as a mixture of perfect fluids $N = 1, 2, \ldots$ with 4-velocity fields U^N_μ and total energy momentum tensor given by

$$T_{\mu\nu} = \sum_N T^N_{\mu\nu} = \sum_N [-p_N g_{\mu\nu} + (\rho_N + p_N) U^N_\mu U^N_\nu]. \quad (1)$$

The components of T^μ_ν are the following:

$$T^0_0 = \sum_N \rho_N \equiv \rho_{\rm T}, \qquad T^i_j = -\sum_N p_N \delta^i_j \equiv -p_{\rm T}\delta^i_j, \quad (2)$$

where $\rho_{\rm T}$ and $p_{\rm T}$ are the total energy density and pressure in the comoving frame $(U^0_N, U^i_N) = (1, 0)$, respectively. Consider now the covariant local conservation law for the mixture, $\nabla_\mu T^{\mu\nu} = 0$. This expression can be worked out explicitly from (1), and then we can contract the result with U^N_ν and use the relation $U^N_\nu \nabla_\mu U^\nu_N = 0$ (which follows immediately from the fact that for any four-velocity vector, we have $U^\mu_N U^N_\mu = 1$). The final result reads [57]

$$\sum_N [U^\mu_N \nabla_\mu \rho_N + (\rho_N + p_N)\nabla_\mu U^\mu_N] = 0. \quad (3)$$

This equation is the local conservation law in a more explicit form, but we can still further reduce it. In the case of a Friedmann-Lemaître-Robertson-Walker (FLRW) metric, it is straightforward to check that for a comoving frame ($U^\mu_N = \delta^\mu_0$), one finds:

$$\nabla_\mu U^\mu_N = 3H \qquad (N = 1, 2, \ldots), \quad (4)$$

and this relation implies that Eq. (3) boils down to

$$\sum_N [\dot\rho_N + 3H(\rho_N + p_N)] = 0. \quad (5)$$

This is the overall conservation law of the fluid mixture in its final and useful form [57].

Up to this point we did not specify the nature of the fluids involved. Let us now assume that we have a mixture of two fluids, matter and vacuum energy. The matter fluid itself is in general a mixture of relativistic matter (i.e., radiation, ρ_r) and nonrelativistic matter (i.e., cold matter, ρ_m) components, but for simplicity we address here a situation in which there is a single matter component that dominates. This component can either be ρ_r (in the early universe after inflation) or ρ_m (well after equality). However, when we discuss a generic epoch we shall denote by ρ the density for the (dominant) matter component or ω-fluid, whatever it be (radiation or cold matter) and by ρ_Λ the vacuum energy density, where $\rho_\Lambda = \Lambda/(8\pi G)$ in natural units. The corresponding pressures for matter and vacuum energy are indicated by P and P_Λ, respectively. The equations of state of the two fluids are: $P = \omega\rho$ and $P_\Lambda = -\rho_\Lambda$ (i.e., $\omega_\Lambda = -1$), where the equation of state

(EoS) parameter for the ω-fluid is a positive constant for a spatially flat FLRW metric. In our case, $\omega = 1/3$ for dominant relativistic matter (i.e., when $\rho = \rho_r$) and $\omega = 0$ for dominant cold matter ($\rho = \rho_m$). The corresponding Einstein field equations of the system formed by a dominant matter component and the vacuum fluid read

$$8\pi G\rho_{\mathrm{T}} \equiv 8\pi G\rho + \Lambda = 3H^2 \tag{6}$$

$$8\pi G p_{\mathrm{T}} \equiv 8\pi G P - \Lambda = -2\dot{H} - 3H^2, \tag{7}$$

where $H \equiv \dot{a}/a$ is the Hubble rate, $a = a(t)$ is the scale factor, and the overdot denotes derivative with respect to the cosmic time t. Let us note that if we consider the two Eqs. (6) and (7) together with the overall conservation law (5), only two of them are independent. For example, if we take the above pair as the two independent equations, then one can easily show that (5) is just a first integral of the system. However, for convenience we may also be interested in using, say, Eq. (6) and the overall conservation law (5). These two are also independent. It should then be clear that any two of the three equations contain all the information and the third one is identically satisfied.

Let us now discuss the possibility, in contrast to ΛCDM case, that Λ is not constant but a function of the cosmic time, i.e., $\rho_\Lambda = \rho_\Lambda(t)$. This is perfectly allowed by the cosmological principle embodied in the FLRW metric. The EoS for the vacuum and matter fluids can still be $P_\Lambda(t) = -\rho_\Lambda(t)$ and $P/\rho = \omega$, respectively, where the latter takes the aforementioned values in the relativistic and nonrelativistic regimes. It is important to realize that under these conditions the above Eqs. (5)–(7) stay formally the same, as it is easy to check. Therefore, applying the conservation law (5) for a dominant matter ω-fluid plus a time-evolving vacuum ($\omega_\Lambda = -1$), we find:

$$\dot{\rho}_\Lambda + \dot{\rho} + 3(1+\omega)\rho H = 0. \tag{8}$$

This law is a consequence of imposing the covariant conservation of the total energy density of the combined system of matter and vacuum, and therefore is a direct reflection of the Bianchi identity satisfied by the geometric side of the Einstein's equations. Such law will play an important role in our discussions. In the ΛCDM model, where $\rho_\Lambda = \mathrm{const}$, it is obvious that it boils down to the standard matter conservation law $\dot{\rho} + 3(1+\omega)\rho H = 0$.

III. GENERAL ANSATZ FOR THE EVOLVING VACUUM AS A FUNCTION OF H

Our main aim in this paper is to study a relevant class of time-evolving models for the vacuum energy. However, we do not aim at an arbitrary function of the cosmic time $\Lambda = \Lambda(t)$. In fact, we focus on a dynamical CC term, Λ, whose primary dependence is on the Hubble rate and from here the vacuum energy inherits its time dependence: $\Lambda(t) = \Lambda(H(t))$. As we will see, this is more in consonance with the expectations in QFT. Nonetheless not all possible functional dependences on H are allowed. In order to obtain a definite decaying Λ cosmology we need to find a viable expression for Λ in terms of the Hubble rate. The motivation for a well-motivated function $\Lambda = \Lambda(H)$ can be provided from different points of view. Let us start from a general phenomenological one, and only afterwards (see the next section) we will motivate it in more formal terms. The existence of two fluid components means that we may introduce the following ratio:

$$\beta(t) = \frac{\rho_\Lambda - \rho_{\Lambda 0}}{\rho + \rho_\Lambda}, \tag{9}$$

where $\rho_{\Lambda 0}$ is a constant vacuum density defining the fiducial constant Λ. This $\beta(t)$ parameter quantifies the time variation of the vacuum energy density. It has the following properties: (i) If $\rho_\Lambda = \rho_{\Lambda 0}$, then $\beta = 0$, and the model is ΛCDM, (ii) If $\rho_{\Lambda 0} = 0$, then the ratio (9) defines a fraction of the vacuum to the total density. If this fraction is constant in the course of the cosmic evolution we have

$$\rho_\Lambda = \beta\rho_{\mathrm{T}}, \tag{10}$$

or, equivalently, from Eq. (6),

$$\Lambda = 3\beta H^2. \tag{11}$$

This kind of model was discussed long ago by many authors [17–19]. It needs only the assumption that the ratio (9) remains constant. However, when confronted with the current observations it provides a poor fit [37]. As a matter of fact, it is ruled out by an even more fundamental reason, because in these models there does not exist a transition redshift from deceleration to acceleration as required by supernovae data. The ansatz (11) implies that the universe is always accelerating or decelerating depending on the value of β. A brief discussion on this point is presented at the end of Sec. VII, see also [39] for a more detailed discussion. More recently, this $\Lambda(H)$-law has also been applied to discuss the late stages of the gravitational collapse [58].

If we, instead, consider that the ratio given by (9) is constant, then we have

$$\Lambda(t) = c_0 + 3\beta H^2(t), \tag{12}$$

where $c_0 = 8\pi G\rho_{\Lambda 0}$. Notice that the present value of the CC in this framework reads $\Lambda_0 = c_0 + 3\beta H_0^2$. Such a model was first proposed in [29] from the point of view of the RG and it has been studied extensively in the literature, cf. Refs. [32,37–39,48]. In contrast to (11) the presence of the additive term is well motivated within the RG approach (see Sec. IV) and allows the existence of a transition from deceleration to acceleration, and of course then also a smooth connection with the ΛCDM model is possible in the limit $\beta \to 0$. Notice that, in contrast, the model (11) has no ΛCDM limit. In general the ratio (9) may not remain constant during the evolution, i.e.,

β should be a time-dependent quantity. In this case the vacuum energy density reads

$$\rho_\Lambda = \rho_{\Lambda 0} + \beta(t)\rho_{\rm T}, \tag{13}$$

or equivalently

$$\Lambda(t) = c_0 + 3\beta(t)H^2. \tag{14}$$

Since $\beta(t)$ is now variable, the value of the current CC is $\Lambda_0 = c_0 + 3\beta(t_0)H_0^2$, where t_0 is the present cosmic time. Let us assume that we can expand the time-dependent parameter $\beta(t)$ as follows: $\beta(t) = \nu + \alpha(\frac{H}{H_I})^n$, where ν, α and H_I are constants whose interpretation will become apparent later on, and n is typically a positive integer $n \geq 1$. The expansion of $\beta(t)$ in this form can be seen as a constant term plus a time-dependent term. The latter should naturally depend on a power of the expansion rate, $n = 1$ being the simplest possibility (although other constraints could change this option). Several aspects of the case $n = 1$ with a flat geometry were discussed long ago in [22], and, later on, the case for closed and hyperbolic geometries was also investigated [23]. The case with $c_0 = 0$ and $\nu = 1 - \beta$ and arbitrary values of n was first phenomenologically proposed in [52] while the case $n = 2$ with $c_0 = \beta = 0$ (plus a linear term in H) was more recently investigated in [53]. In general, for the above $\beta(t)$ we arrive at the general ansatz:

$$\Lambda(H) = c_0 + 3\nu H^2 + 3\alpha\frac{H^k}{H_I^{(k-2)}}, \tag{15}$$

where $k = n + 2$. It is interesting to note that the next-to-leading higher order power, i.e., the case $k = 4$, can be motivated on more fundamental QFT grounds, as shown long ago in [59] and more recently in [30] within the framework of the modified anomaly-induced inflation scenarios. These are a generalization of Starobinsky's model type of inflation [60], in which the vacuum effective action for massive quantum fields can be computed using the conformal representation of the fields action [61].

If we would not attend other considerations, the integer k in Eq. (15) is generally unrestricted, apart from $k \geq 3$. Obviously the case $k = 2$ (i.e., $n = 0$) is not considered because it corresponds to the situation $\beta = \nu + \alpha =$ const, considered in the original RG formulation (12) (see next section) which already contains H^2 as the highest power of the Hubble rate. This situation is equivalent to $k = 0$ upon redefining c_0 and with $\beta = \nu =$ const. Nontrivial departure of these cases thus requires $k \geq 3$.

The constant additive term in (15) obviously represents the dominant contribution at very low energies (i.e., when $H \approx \mathcal{O}(H_0) \ll H_I$). The H^2 term represents a small correction (if $\nu \ll 1$) to the dominant term at the present time. While it provides a mild time-evolving behavior to the vacuum energy density at intermediate times. On the other hand, the H^k ($k \geq 3$) power acquires a great relevance in the early universe, near the H_I energy scale—interpreted as the inflationary expansion rate.

Since H_I is presumably large, it is clear that $\beta(t_0) \simeq \nu$ for any n and hence the value of the CC today is essentially $\Lambda_0 = c_0 + 3\nu H_0^2$ for all models of the class (15). Thus, effectively, for any $k \geq 3$ the proposed model (15) is very close to the model (12) for a description of the postinflationary cosmology, including of course the evolution near the current time. It follows that the coefficient ν is the relevant one for the dynamical evolution of the vacuum energy in most of the universe's history. However, for the early universe the additional term H^k takes over and the effective behavior of Eq. (15) is then $\Lambda(t) \simeq 3\alpha H^k(t)/H_I^{(k-2)}$ ($k \geq 3$), and here the relevant coefficient is α together with the inflationary scale H_I. Of course α and H_I appear to be a convenient way to break down the single coefficient of the dominant power H^k. To disentangle the value of the dimensionless coefficient α we would need to relate H_I to some physical high-energy scale, for example a typical grand unified theory (GUT) scale associated to the inflationary time.

Let us mention that the covariance of the effective action of QFT in curved space-time indicates that the even powers of H are preferred (see the next section); in other words, the new term H^k correcting the original expression (12) is expected to have $k = 2m$ (with $m = 2, 3, \ldots$). Naturally, despite the fact that the odd powers $k = 3, 5, \ldots$ in (15) are not favored, we will not completely neglect them, if only from the phenomenological point of view (see Refs. [22,23]). In contrast, the case $k = 0$ leads to the model (12) considered in [29], which is adequate for the more recent universe [32,37,38] but not for the very early stages. In this paper, we are proposing a generalization that leads to the unification model (15) for the complete description of the cosmological history from the very early universe to the present time.

Now, combining Eqs. (6), (8), and (15), and using the EoS of the fluid components we obtain the following differential equation for the time evolution of the Hubble parameter:

$$\dot{H} + \frac{3}{2}(1+\omega)H^2\left[1 - \nu - \frac{c_0}{3H^2} - \alpha\left(\frac{H}{H_I}\right)^n\right] = 0. \tag{16}$$

Remarkably there are two constant value solutions to this equation, namely $H = H_I[(1-\nu)/\alpha]^{1/n}$, corresponding to the very early universe, i.e., when $c_0 \ll H^2$. On the other hand, at late times, when $H \ll H_I$ we have $H = [c_0/3(1-\nu)]^{1/2}$, whereby $\Lambda \approx c_0$ which behaves as an effective cosmological constant. Also using Eq. (16) the deceleration parameter $q \equiv -\ddot{a}a/\dot{a}^2 = -1 - \dot{H}/H^2$ is given by

$$q(H) = \frac{3}{2}(1+\omega)\left[1 - \nu - \frac{c_0}{3H^2} - \alpha\left(\frac{H}{H_I}\right)^n\right] - 1. \tag{17}$$

We shall present below the various phases of the decaying vacuum cosmology (15), starting from an unstable inflationary phase powered by the huge value H_I presumably connected to the scale of a GUT or even the Planck scale M_P, then it deflates (with a massive production of relativistic particles), and subsequently evolves into the standard radiation and matter eras. Finally, it effectively appears today as a slowly dynamical dark energy.

IV. RUNNING VACUUM $\Lambda = \Lambda(H)$

In the previous section we have motivated the time evolution of the vacuum energy density as a function of the Hubble rate using a general phenomenological argumentation. However, the running of the vacuum energy is expected in QFT in curved space-time on more fundamental grounds [29–31], see also [35,36] and references therein. Running couplings in flat QFT provide a useful theoretical tool to investigate theories as QED or QCD, where the corresponding gauge coupling constants run with a scale μ associated to the typical energy of the process, $g = g(\mu)$. Similarly, in the effective action of QFT in curved space-time ρ_Λ should be an effective coupling depending on a mass scale μ. In the universe we should expect that the running of ρ_Λ from the quantum effects of the matter fields is associated with the change of the space-time curvature, and hence with the change of the typical energy of the classical gravitational external field linked to the FLRW metric. As this energy is pumped into the matter loops from the tails of the external gravitational field, it could be responsible for the physical running. Therefore we naturally associate μ^2 to R, where (for flat FLRW metric)

$$|R| = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 12H^2 + 6\dot{H}. \tag{18}$$

It follows that μ^2 is in correspondence with H^2 and $\dot{H}$. For simplicity we concentrate on the setting $\mu = H$ as we expect that it already captures the essential dynamics (see [39]). Within this RG approach the rate of change of ρ_Λ with $\mu = H$ should satisfy a corresponding RG equation:

$$\begin{aligned}(4\pi)^2\frac{d\rho_\Lambda(\mu)}{d\ln\mu^2} &= \sum_{m=1,2,\ldots} A_{2m}\mu^{2m} \\ &= A_2\mu^2 + A_4\mu^4 + A_6\mu^6\cdots \end{aligned}\tag{19}$$

The right-hand side of this expression defines essentially the β-function for the RG running of ρ_Λ. The coefficients A_{2m} receive loop contributions from boson and fermion matter fields of different masses M_i. Notice that only the even powers of $\mu = H$ are involved, since in this formulation $\rho_\Lambda(H)$ is of course part of the effective action of QFT in curved space-time and hence it should be a covariant quantity [29–31]. Worth noting is that we have omitted the A_0 term in (19), as it would be of order M_i^4 and hence would trigger a too fast running of ρ_Λ. This can also be formally justified from the fact that all known particles satisfy $\mu < M_i$ (for $\mu = H$). Thus, since none of them is an active degree of freedom for the running of ρ_Λ, only the subleading terms are available. The first subleading term is the $A_2\mu^2$ one, where A_2 has dimension of mass squared, namely $A_2 \sim \sum_i a_i M_i^2$ where the sum is over the masses of all fields involved in the computation of the β-function (including their multiplicities). Similarly, since all the coefficients A_{2m} (except A_4) are dimensionful, it is convenient to rewrite them appropriately in a way such that the mass dimensions are explicit. Thus we rewrite (19) as follows:

$$\frac{d\rho_\Lambda(\mu)}{d\ln H^2} = \frac{1}{(4\pi)^2}\sum_i\left[a_i M_i^2 H^2 + b_i H^4 + c_i\frac{H^6}{M_i^2} + \cdots\right]. \tag{20}$$

The sum over the masses of the fields involved in the loop contributions is now explicit. Specific realizations of the structure (20) can be obtained in one-loop calculations within particular frameworks, see e.g., [30]. As we can see, the series became now an expansion in powers of H. If we integrate equation (20) to obtain $\rho_\Lambda(H)$, an additive term (independent of H) obviously appears as well. In other words, the result for $\Lambda(H) = 8\pi G\rho_\Lambda(H)$ nicely adapts to the form (15) suggested by the general argument of the previous section, which means that the RG formulation may provide a fundamental link of that form with QFT in curved space-time. However, as emphasized, only the even powers of H are involved in the RG realization, owing to the general covariance of the effective action. As it is obvious, the expansion (20) converges very fast at low energies, where H is rather small—certainly much smaller than any particle mass. No other H^{2m}-term beyond H^2 (not even H^4) can contribute significantly on the right-hand side of Eq. (20) at any stage of the cosmological history below the GUT scale M_X, typically a few orders of magnitude below the Planck scale $M_P \sim 10^{19}$ GeV.

However, if we wish to have access to the physics of inflation and in general to the very early states of the cosmic evolution, we need to keep at least the terms H^4. It is interesting to note the structure of the leading term in the series (20), i.e., $\sim\sum_i M_i^2 H^2$. This term is of course dominated by the loop contributions of the heaviest fields with masses M_i of order of M_X, the GUT scale near the Planck mass. It follows that in the early universe (when H is also close, but below, $M_i \sim M_X$) the H^4 effects can also be significant, whereas the terms H^6/M_i^2 and above are less and less important. Therefore, the dominant part of the series (20) is expected to be naturally truncated at the H^4 term. These terms should contain the bulk of the high energy contributions within QFT in curved space-time, namely within a semiclassical description of gravity near but (possibly a few orders) below the Planck scale. Models of inflation based on higher order terms inspired by the RG framework have existed for a long time in the literature, see [59] as well as the unified inflation-dark energy framework of [30] (see also [22,23,52,53] for a more phenomenological treatment).

We can find the explicit relation between the one-loop coefficients of the RG equation (20) with the phenomenological coefficients introduced in Sec. III. Let us consider the case $n = 2$, for which the highest power of the Hubble rate in the vacuum expression is H^4. Upon integrating the RG equation (20) and comparing with Eq. (15) we obtain

$$\nu = \frac{1}{6\pi} \sum_{i=f,b} c_i \frac{M_i^2}{M_P^2}, \tag{21}$$

and

$$\alpha = \frac{1}{12\pi} \frac{H_I^2}{M_P^2} \sum_{i=f,b} b_i. \tag{22}$$

A few words will help to better interpret this result. First of all let us note that ν acts indeed as the reduced (dimensionless) β-function for the RG running of ρ_Λ at low energies, whereas α plays a similar role at high energies. Moreover, both coefficients are predicted to be naturally small because $M_i^2 \ll M_P^2$ for all the particles, even for the heavy fields of a typical GUT. In the case of the low energy coefficient ν, a concrete realization of the structure (21) is given in [30], and an estimate within a generic GUT is found in the range $|\nu| = 10^{-6}$–10^{-3}. Similarly, the dimensionless coefficient α is naturally predicted small, $|\alpha| \ll 1$, because the inflationary scale H_I is certainly below the Planck scale M_P. In a typical GUT where $M_X \sim 10^{16}\ \mathrm{GeV}^4$ we have $H_I/M_P \sim M_X^2/M_P^2 \lesssim 10^{-6}$. Even counting the large multiplicities of the fields in usual GUT's, the two coefficients ν and α are expected to be rather small, which is indeed the natural expectation since they play the role of one-loop β-functions at the respective low and high energy scales. Using a joint likelihood analysis of the recent supernovae type Ia data, the CMB shift parameter, and the baryonic acoustic oscillations, one finds that the best fit value for ν in the case of a flat universe is at most of order $|\nu| = \mathcal{O}(10^{-3})$ [37,38], which is nicely in accordance with the aforementioned theoretical expectations.

V. FROM THE EARLY DE SITTER STAGE TO THE RADIATION PHASE

Let us first discuss the transition from the initial de Sitter stage to the radiation phase, while $c_0 \ll H^2$. The solution (A2)—see the Appendix—of Eq. (16) for $\omega = 1/3$ and $c_0 = 0$ becomes

$$H(a) = \frac{\tilde{H}_I}{[1 + Da^{2n(1-\nu)}]^{1/n}}, \tag{23}$$

where we have defined $\tilde{H}_I \equiv (\frac{1-\nu}{\alpha})^{1/n} H_I$, is the critical Hubble parameter associated to the initial de Sitter era, or

$$\int_{a_\star}^{a} \frac{d\tilde{a}}{\tilde{a}} [1 + D\tilde{a}^{2n(1-\nu)}]^{1/n} = \tilde{H}_I t, \tag{24}$$

where t here is the time elapsed after (approximately) the end of the inflationary period, indicated by $t_\star$, and we have defined $a_\star = a(t_\star)$. The integration constant D is fixed from the condition $H(a_\star) \equiv H_\star$, thus

$$D = a_\star^{-2n(1-\nu)} \left[\left(\frac{\tilde{H}_I}{H_\star} \right)^n - 1 \right]. \tag{25}$$

Equation (24) will be useful below for particular considerations. However, rather than directly integrating this equation it is possible to retake (23) and cast it in a more appropriate form that allows us to express the result $t = t(a)$ in terms of special functions. This is done in the Appendix. The final result is

$$t(a) = \frac{(1 + Da^{2n(1-\nu)})^{\frac{1+n}{n}}}{2(1-\nu)\tilde{H}_I D a^{2n(1-\nu)}} \times F\left[1, 1, 1 - \frac{1}{n}, \frac{-1}{Da^{2n(1-\nu)}} \right], \tag{26}$$

where $F[\alpha_1, \alpha_2, \alpha_3, z]$ is the Gauss hypergeometric function, and as in (24) we count the time passed after $t_\star$, i.e., t is the cosmic time within the FLRW regime. Using the Einstein equations and the above solutions we can obtain the corresponding vacuum, radiation, and total energy densities:

$$\rho_\Lambda(a) = \tilde{\rho}_I \frac{1 + \nu D a^{2n(1-\nu)}}{[1 + Da^{2n(1-\nu)}]^{1+2/n}}, \tag{27}$$

$$\rho_r(a) = \tilde{\rho}_I \frac{(1-\nu) D a^{2n(1-\nu)}}{[1 + Da^{2n(1-\nu)}]^{1+2/n}}, \tag{28}$$

$$\rho_{\mathrm{T}}(a) = \tilde{\rho}_I \frac{1}{[1 + Da^{2n(1-\nu)}]^{2/n}}. \tag{29}$$

where we have defined

$$\tilde{\rho}_I \equiv \frac{3\tilde{H}_I^2}{8\pi G} \tag{30}$$

is the primeval critical energy density associated with the initial de Sitter stage. We can see from (27) that the value (30) just provides the vacuum energy density for $a \to 0$, namely $\rho_\Lambda(0) = \tilde{\rho}_I$. As $|\nu| \ll 1$ we have $\tilde{\rho}_I/\rho_I \sim \alpha^{-2/n}$ and hence the density $\tilde{\rho}_I$ can differ a few orders of magnitude from ρ_I since we also expect (see the previous section) that $|\alpha| \ll 1$. Let us also emphasize from the previous formulas that for $a \to 0$ we have $\rho_r/\rho_\Lambda \propto a^{2n(1-\nu)} \to 0$, i.e., the very early universe is indeed vacuum dominated with a negligible amount of radiation. For the numerical analysis of the energy densities, see Fig. 1.

Notice that the constant (25) entering Eq. (23) is greater than zero precisely for $\tilde{H}_I > H_\star$, which is tantamount to say $\rho_\star < \tilde{\rho}_I$, where $\rho_\star \equiv 3H_\star^2/8\pi G$ is the critical energy density at the time $t = t_\star$. The existence of a point marking the decrease of the energy density from the initial steady

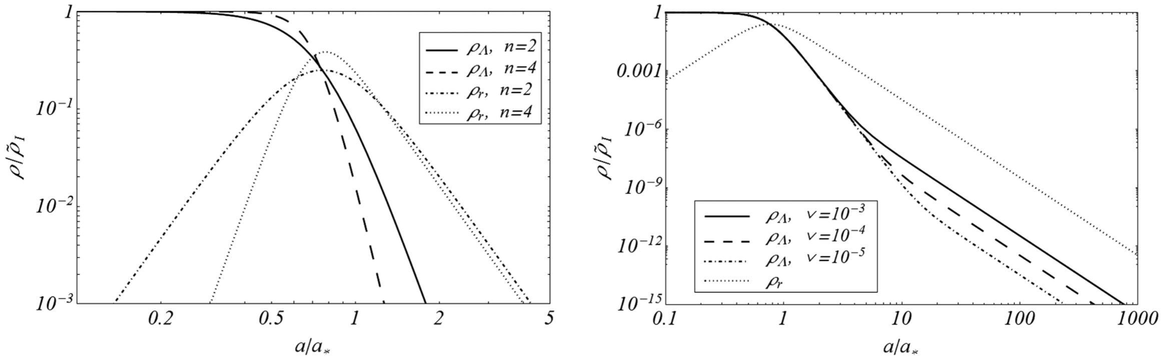

FIG. 1. Left panel: The evolution of the vacuum and radiation energy densities during the primordial era, where $H^2 \gg c_0$. We normalize the densities with respect to the primeval critical value $\tilde{\rho}_I$ defined in (30). The plots show that the decay of the vacuum density, as well as the production and subsequently dilution of radiation, occur in a faster way for large values of the parameter n [recall that $k = n + 2$ in Eq. (15)], thereby ensuring the universality of the graceful exit for any $n \geq 2$. Additionally, in this figure we can see that the vacuum density always decays faster than it does the radiation density after the transition period. Right panel: The behavior of the vacuum density with the variation of the parameter ν for $n = 2$. In this graph, we can see that during the radiation-dominated era the vacuum density ceases to decay; it only dilutes with time (in a similar way as the radiation energy density) due to the effect of the expansion. The precise instant when this change occurs is earlier for larger values of the parameter ν. On the other hand, the evolution of the radiation energy density is affected very little by the variation of the parameter ν, for $\nu \leq 10^{-3}$. In this figure we show the radiation energy density for $\nu = 10^{-4}$.

value $\tilde{\rho}_I$ is indeed the condition that points to a deflationary period after inflation.

For $Da^{2n(1-\nu)} \ll 1$ (during the very early universe) the solution (23) can be approximated by the constant value solution $H \approx \tilde{H}_I$. As mentioned, the vacuum energy density remains almost constant $\rho_\Lambda \approx \tilde{\rho}_I$ in this period and coexists with a negligible radiation density, which just starts to grow as $\rho_r \simeq \rho_\Lambda(1-\nu)Da^{2n(1-\nu)}$. This stage obviously depicts the primeval de Sitter era in the cosmic evolution, with

$$a(t) \propto \exp[\{\tilde{H}_I t\}], \qquad (31)$$

in which the universe undergoes a process of primordial inflation. The result (31) can be derived by expanding the solution (26) around $Da^{2n(1-\nu)} \ll 1$:

$$\tilde{H}_I t \approx \frac{1}{2n(1-\nu)} \times [C + \ln Da^{2n(1-\nu)}], \qquad (32)$$

where C is a constant (dependent on n) not playing a role in this argument. Notice that Eq. (31) can also be substantiated by simply letting $a \to 0$ before integrating Eq. (24).

The outcome of the above considerations is that for $D \neq 0$ the universe starts without a singularity and thus this model overcomes the horizon problem. The universe then evolves naturally toward a radiation-dominated universe (hence providing a useful clue to explaining the "graceful exit" from the inflationary stage, see Fig. 2). On the other hand, a light pulse beginning at $t = -\infty$ will have traveled by the cosmic time t a physical distance, $d_H(t) = a(t)\int_{-\infty}^{t} \frac{dt'}{a(t')}$, which diverges thereby implying the absence of particle horizons, thus the local interactions may homogenize the whole universe.

The solution for the radiation energy density (28) reaches a maximum value when the scale factor a takes on the value $a_* \equiv (2D/n)^{-1/2n(1-\nu)}$, which is the value when the inflation period is accomplished and the radiation-dominated era begins (see Figs. 1 and 2).

For $Da^{2n(1-\nu)} \gg 1$ the solution (23) can be approximated as

$$H \approx \tilde{H}_I D^{-1/n} a^{-2(1-\nu)}, \qquad (33)$$

which displays the behavior $H \sim a^{-2(1-\nu)} \sim a^{-2}$ in the limit of small $|\nu|$. Similarly from (26) we find

$$t \approx a^{2(1-\nu)}. \qquad (34)$$

The derivation of the latter expression is particularly straightforward from (24) if we use the limit $Da^{2n(1-\nu)} \gg 1$ before integration. As $|\nu| \ll 1$, it is obvious that we have essentially reached the radiation domination era for which $a \propto t^{1/2(1-\nu)} \simeq t^{1/2}$. This is confirmed after inspecting the radiation density (28), which decays as $\rho_r \propto (1-\nu)a^{-4(1-\nu)} \sim a^{-4}$. We can also see from (27) that the vacuum energy density follows a similar decay law $\rho_\Lambda \propto \nu a^{-4(1-\nu)}$, but is suppressed by the factor $\rho_\Lambda/\rho_r \propto \nu$ (with $|\nu| \ll 1$) as compared to the radiation density. This is exactly the opposite situation to the very early period when $Da^{2n(1-\nu)} \ll 1$, in which the vacuum energy density is huge and stuck at the value $\tilde{\rho}_I$, whereas the radiation density is largely suppressed by the power $a^{2n(1-\nu)}$ of the very small scale factor at that time. In between these two eras, we see that we can have either huge relativistic particle production $\rho_r \propto a^{2n(1-\nu)}$ in the deflation period (namely around $Da^{2n(1-\nu)} < 1$) or

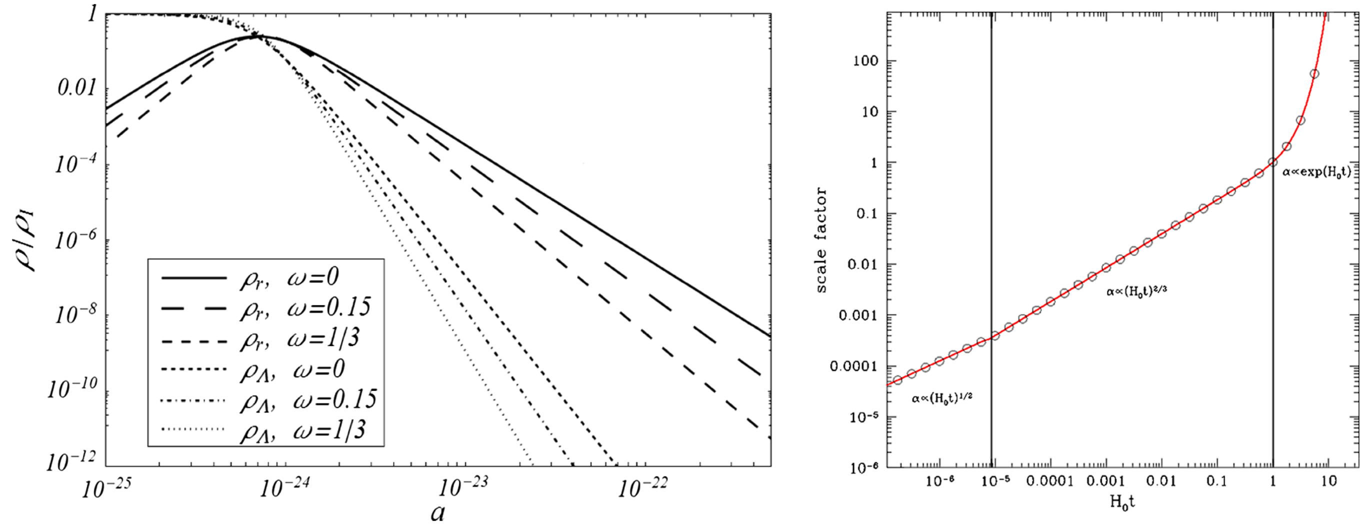

FIG. 2 (color online). Left panel: Universality of the graceful exit with respect to the variation of the EoS parameter for the matter ω-fluid. Once more we normalize the densities with respect to the primeval critical value $\tilde{\rho}_I$ defined in (30). This figure shows that the rate of the vacuum decay, radiation production, and subsequent dilution is larger for greater values of ω. As always, the vacuum decay is faster than the rate of dilution of the radiation, thereby ensuring the transition toward a radiation domination era after the end of the inflationary stage. Right panel: The evolution of the scale factor predicted by the decaying $\Lambda(H)$ model at the late stage $H_I \gg H$ for $\nu \sim 10^{-3}$ (solid line) versus the traditional ΛCDM cosmology (open points). In this plot we have adopted the best fit, $\Omega_\Lambda^0 = 0.6825$, from the recent results of PLANCK data [11]. Clearly, the expansion history of the scale factor of the $\Lambda(H)$ model is almost indistinguishable from the ΛCDM model for the entire postinflationary era up to our days, and into the future.

standard dilution $\rho_r \propto a^{-4}$ (up to small corrections of order $|\nu| \ll 1$) well in the radiation era ($Da^{2n(1-\nu)} \gg 1$).

A. Radiation temperature

In the case of $\omega = 1/3$, we have relativistic matter production $\rho_r \propto a^{2n(1-\nu)}$ during the deflationary era and corresponding dilution $\rho_r \propto a^{-4(1-\nu)}$ during the subsequent radiation-dominated era (due to the expansion of the universe). Considering "adiabatic" expansion of the universe during both eras, the radiation energy density scales as $\rho_r \sim T_r^4$ [62,63] with its temperature. Thus we can see that the radiation temperature grows as $T_r \propto a^{n(1-\nu)/2}$ during the initial de Sitter era of accelerated expansion if the specific entropy per particle remains constant during this period. Hence, the universe is naturally heated before it enters the radiation-dominated era.

After the de Sitter stage, the temperature decreases continuously in the course of the expansion as $T_r \propto a^{-(1-\nu)}$, namely very close to $1/a$ for $|\nu| \ll 1$, as it should be for an noninteracting adiabatic expansion. Accordingly, the comoving number density of photons scales as $n_\gamma \sim T^3 \propto a^{-3(1-\nu)}$, which shows a tiny departure from the ΛCDM case since the vacuum energy density itself is evolving mildly owing to the nonzero value of ν. For further interesting thermodynamical considerations about this type of universes (starting from a de Sitter phase) which show their viability from the point of view of the generalized second principle of thermodynamics [64]. It is worth noting that several models starting from a de Sitter phase (deflation) induced by gravitational particle creation of relativistic particles have also been discussed in the literature [65]. As occurs in the present scenario, some of them also evolve between two extreme de Sitter phases [66] and the general thermodynamic analysis presented in [64] remains valid (see also [67] for a possible connection between scenarios driven by decaying Λ models and gravitationally induced particle creation).

B. Primordial transition: From an accelerating vacuum to a decelerating radiation phase

In this case ($c_0 \ll H^2$ and for $\omega = 1/3$) the deceleration parameter follows from (17) and (23):

$$q(a) = \frac{(1-2\nu)Da^{2n(1-\nu)} - 1}{Da^{2n(1-\nu)} + 1}. \tag{35}$$

It varies from $q_I \approx -1$ when $a \to 0$ to a positive value near the standard radiation regime ($q = 1 - 2\nu \simeq 1$) when $Da^{2n(1-\nu)} \gg 1$. The primordial transition (pt) between the early accelerating period and the decelerating radiation phase (when still $H \gg H_0$) occurs for the scale factor:

$$a_{\rm pt} = \left[\frac{1}{(1-2\nu)D}\right]^{1/2n(1-\nu)}. \tag{36}$$

In Fig. 3 we display some numerical examples of the evolution of $q(a)$ in this period.

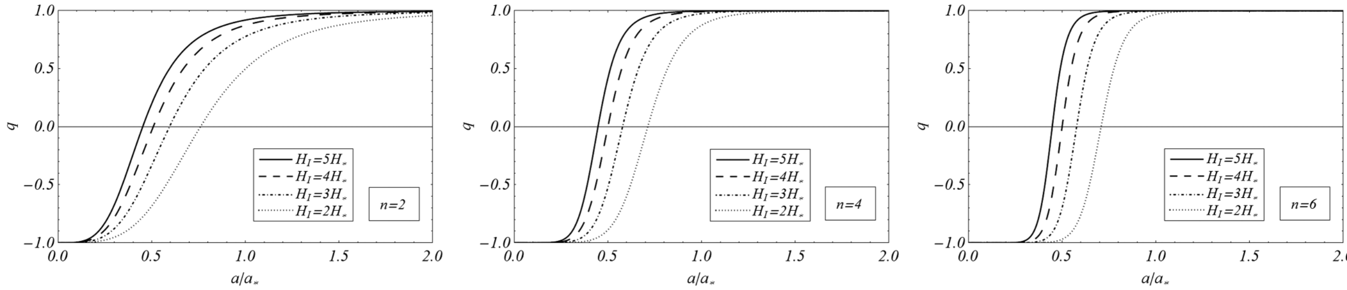

FIG. 3. Evolution of the decelerating parameter during the primordial era for several values of the free parameter n. All plots were obtained for $H^2 \gg c_0$ and show the universality of the transition between the early accelerated de Sitter stage ($q \simeq -1$) and the subsequent decelerated radiation era ($q \simeq 1$) as driven by the H^{n+2} decaying vacuum models (from the left to the right we have fixed, respectively, $n = 2, 4, 6$; equivalently, $k = 4, 6, 8$ in Eq. (15). Note that the transition occurs faster for the bigger values of the inflationary energy scale H_I and n. This general behavior does not change appreciably for any finite value of $n \geq 1$.

VI. ALTERNATIVE DESCRIPTION IN TERMS OF THE EFFECTIVE POTENTIAL

In Sec. IV we have elaborated on the motivation of the present model within the general structure of the effective action of QFT in curved space-time, and we have used the RG equation (20) which naturally leads to the expression of the unified model of the vacuum energy density, Eq. (15). Although at the moment we cannot provide the effective action leading to this kind of framework in the general case [31], except in some particular formulations [30], we can mimic it through an effective scalar field (ϕ) model [68]. Let us note that any time-evolving vacuum energy density model can be described in this way [34]. This can be useful for the usual phenomenological descriptions of the DE, and can be obtained from the usual correspondences: $\rho_T \to \rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_T \to p_\phi = \dot{\phi}^2/2 - V(\phi)$ in Friedmann's Eqs. (6) and (7). We find $4\pi G\dot{\phi}^2 = -\dot{H}$ and

$$V_{\rm eff}(a) = \frac{3H^2}{8\pi G}\left(1 + \frac{\dot{H}}{3H^2}\right) = \frac{3H^2}{8\pi G}\left(1 + \frac{1}{3}\frac{d\ln H}{d\ln a}\right). \quad (37)$$

The effective potential can be readily worked out for our model starting from the expression of the Hubble function in the early universe (23). We perform the calculation neglecting the small $\mathcal{O}(\nu)$ corrections, as they are not important for the present discussion. The final result is the following:

$$V_{\rm eff}(a) = \frac{\rho_I}{\alpha^{2/n}}\frac{1 + Da^{2n}/3}{(1 + Da^{2n})^{(n+2)/n}}, \quad (38)$$

where $\rho_I \equiv 3H_I^2/8\pi G$. The interesting case $n = 2$, corresponding to having a term H^4 in the high energy sector of the vacuum energy density (15), yields

$$V_{\rm eff}(a)|_{n=2} = \frac{\rho_I}{\alpha}\frac{1 + Da^4/3}{(1 + Da^4)^2}. \quad (39)$$

This specific form was first derived in [55,56], and is just a particular case of the general effective potential (38). From the general expression it becomes clear that the potential energy density remains constant, $V_{\rm eff} \sim \rho_I/\alpha$, while $a \ll D^{-1/(2n)}$ (i.e., before the transition from inflation to the deflationary regime). However, when the transition is left well behind (i.e., when $a \gg D^{-1/(2n)}$) the effective potential (38) decreases in the precise form $V(a) \sim a^{-4}$, valid for all n, as it should be in order to describe a radiation-dominated universe independently of the value of n. This result corroborates, in the effective scalar field language, the transition of the de Sitter stage into the relativistic FLRW regime, which we have described previously in the original Einstein picture, and shows once more that our unified model leads to the correct radiation-dominated epoch for any value of n. In other words, the entire class of $\Lambda(H)$ models (15) leads to an acceptable solution of the graceful exit problem.

VII. FROM THE MATTER TO THE RESIDUAL VACUUM DOMINATION

In the following we consider the expanding universe well after the inflationary period and the radiation epoch. To be more precise, we address the universe at a time after recombination, therefore consisting of dust ($\omega = 0$) plus the running vacuum fluid described by (15) with $H \ll H_I$. In this case the H^k term ($k \geq 3$) is completely negligible compared to H^2 and that equation reduces to

$$\Lambda(H) = \Lambda_0 + 3\nu(H^2 - H_0^2), \quad (40)$$

where $\Lambda_0 \equiv c_0 + 3\nu H_0^2$ is the current value of the CC. Obviously, c_0 plays an essential role to determine the value of Λ, whereas the H^2 dependence gives some remnant dynamics even today, which we can use to fit the parameter ν to observations. Using a joint likelihood analysis of the recent supernovae type Ia data, the CMB shift parameter, and the baryonic acoustic oscillations one finds that the best fit parameters for a flat universe are: $\Omega_{m0} \simeq 0.27$–0.28 and $|\nu| = \mathcal{O}(10^{-3})$ (see [37–39]). It is remarkable that the fitted value of ν is within the theoretical expectations when this parameter plays the role of β-function of the running

CC. As already mentioned, in specific frameworks one typically finds $\nu = 10^{-5}$–10^{-3} [30].

For $H \ll H_I$ and $\omega = 0$ the evolution equation of the Hubble parameter (16) becomes simplified. Trading the cosmic time by the scale factor, upon using $d/dt = aHd/da$, it can be rewritten as

$$aHH' + \frac{3}{2}(1-\nu)H^2 - \frac{c_0}{2} = 0, \tag{41}$$

where the prime denotes derivative with respect to the scale factor a. The above equation can now be integrated with the result (A19) (see the Appendix)

$$H^2(a) = \frac{H_0^2}{1-\nu}[(1-\Omega_\Lambda^0)a^{-3(1-\nu)} + \Omega_\Lambda^0 - \nu], \tag{42}$$

where we have used the corresponding boundary condition at the present time: $c_0 = 3H_0^2(\Omega_\Lambda^0 - \nu)$. Notice that the previous equation can, if desired, easily be reexpressed in terms of the redshift z through the relation $1 + z = 1/a$.

Similarly, the matter and vacuum energy densities are found to be (see the Appendix):

$$\rho_m(a) = \rho_m^0 a^{-3(1-\nu)}, \tag{43}$$

and

$$\rho_\Lambda(a) = \rho_\Lambda^0 + \frac{\nu\rho_m^0}{1-\nu}[a^{-3(1-\nu)} - 1]. \tag{44}$$

where ρ_m^0 and ρ_Λ^0 are the corresponding values at present ($a = 1$). The total energy density reads

$$\rho_T(a) = \frac{\rho_m^0}{1-\nu}[a^{-3(1-\nu)} - \nu] + \rho_\Lambda^0. \tag{45}$$

Integrating once more the equation (42) with respect to the cosmic time we obtain the following time dependence of the scale factor:

$$a(t) = \left(\frac{1-\Omega_\Lambda^0}{\Omega_\Lambda^0 - \nu}\right)^{\frac{1}{3(1-\nu)}} \times \sinh^{\frac{2}{3(1-\nu)}}\left[3H_0\sqrt{(1-\nu)(\Omega_{\Lambda 0} - \nu)}\,t/2\right]. \tag{46}$$

As expected, for $\nu \ll 1$ at late enough times the above solution mimics the Hubble function $H(a)$ of the usual flat Λ-cosmology, which means that the final dynamics of the universe is determined by a single parameter namely Ω_Λ^0 or Ω_m^0, which are well known to be related by the cosmic sum rule $\Omega_m^0 + \Omega_\Lambda^0 = 1$.

From these equations it is clear that for $\nu = 0$ we recover exactly the ΛCDM expansion regime, the standard scaling law for nonrelativistic matter and a strictly constant vacuum energy density $\rho_\Lambda = \rho_\Lambda^0$ (hence $\Lambda = \Lambda_0$). Recalling that $|\nu|$ is found to be rather small when the model is confronted with the cosmological data, $|\nu| \leq \mathcal{O}(10^{-3})$ [37,38], we see that the model under consideration deviates very small from the ΛCDM, specially in the postinflationary epoch, where the only distinctive trace left of the model is the existence of a slowly evolving vacuum energy density or cosmological term (40). This is compatible with the general notion of dynamical dark energy, which in this case would be caused by a dynamical vacuum in interaction with matter.

At very late time we get an effective cosmological constant dominated era, $H \approx H_0\sqrt{(\Omega_\Lambda^0 - \nu)/(1-\nu)}$, see Eq. (42) for sufficiently large a, that implies a pure de Sitter phase of the scale factor. This is the late time de Sitter phase or DE epoch.

A. The deceleration parameter in recent times

In the epoch under consideration, we have $\omega = 0$ and $H/H_I \ll 1$. Thus, with the help of Eqs. (17) and (42) the deceleration parameter takes the form

$$q(a) = \frac{(1-3\nu)\Omega_m^0 a^{-3(1-\nu)} - 2(1-\nu-\Omega_m^0)}{2\Omega_m^0 a^{-3(1-\nu)} + 2(1-\nu-\Omega_m^0)}, \tag{47}$$

In the limit $\nu \to 0$ this expression reduces to that of the ΛCDM model. In particular, the current value ($a = 1$) is $q_0 = (3\Omega_m - 2)/2 \simeq -0.58$, where $\Omega_m^0 \simeq 0.28$. The late-time transition (lt)—in contrast to the aforementioned primordial transition (36)—between the decelerated matter-dominated era and the late accelerated residual vacuum stage of the expanding universe occurs when

$$a_{\rm lt} = \left[\frac{(1-3\nu)\Omega_m}{2(1-\nu-\Omega_m)}\right]^{1/3(1-\nu)}. \tag{48}$$

In the limit $\nu \to 0$ it gives $a_{\rm lt} \simeq 0.58$. In Fig. 4 we show this late transition point and compare it with the slightly different values obtained for the case when $\nu \neq 0$.

Despite the dynamical character of the vacuum energy (40) near our time, it is important to understand that a model of this kind would not work for $c_0 = 0$, i.e., with

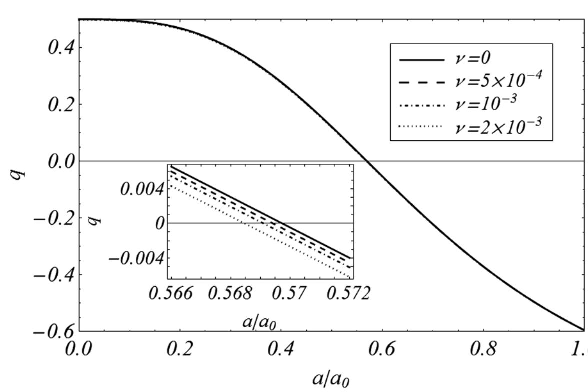

FIG. 4. Evolution of the decelerating parameter during the late stages, when $H \ll H_I$. This figure shows the small departure of the current model (with $\nu \lesssim 10^{-3}$) from the ΛCDM model. The effect of greater values of ν is summarized in shifting forward in time the transition point from deceleration to acceleration into the current accelerated stage.

only pure H-dependent terms on Λ. This has been proven in [37,39] and recently discussed also in [69]. The basic drawback of the $c_0 = 0$ models is that the deceleration parameter never changes sign, and therefore the universe always accelerates or always decelerates [39]. In the present case this can be seen as follows. We can easily check that the condition $c_0 = 0$ enforces Eq. (42) to take the simpler form $H^2(a) = H_0^2 a^{-3(1-\nu)}$. From here we immediately find

$$q = -1 - a\frac{H'(a)}{H(a)} = -1 + \frac{3}{2}(1-\nu). \qquad (49)$$

It follows that acceleration ($q < 0$) is possible only for $\nu > 1/3$, which is unacceptable since we have emphasized that $|\nu| \ll 1$. What is more, since q given by (49) is a constant (i.e., independent of time or of the scale factor) it can only have a sign for a given value of ν, so even if we would admit $\nu > 1/3$ as a mere phenomenological possibility, we would be also admitting that the universe has been accelerating forever, which is of course difficult to accept.

B. The present value of the vacuum energy

After showing the importance of having a nonvanishing c_0 term in our unified vacuum model $\Lambda(H)$, Eq. (15), specially for the low energy segment of the cosmological observations, let us note that the RG formulation of it (cf. Sec. IV) provides a natural explanation for the presence of such $c_0 \neq 0$ value, to wit: the integration of the RG equation (20) must necessarily lead to a nonvanishing additive term in the structure of $\rho_\Lambda(H)$. Therefore, a term of this sort is naturally motivated in this framework. From it the current value of the vacuum energy density reads $\rho_\Lambda^0 = (c_0 + 3\nu H_0^2)/(8\pi G)$. Of course the value of c_0 must be fixed by the boundary condition of the RG differential equation, which is fixed by current observations: $\rho_\Lambda^0 = \rho_\Lambda(H_0)$.

The following observation is now in order: despite our model providing a dynamical explanation for the drastic reduction of the early vacuum energy of our universe from $\rho_\Lambda(H_I)$ to the comparatively very small quantity $\rho_\Lambda^0 \ll \rho_\Lambda(H_I)$, and at the same time ensuring that $\rho_\Lambda(H)$ will be totally harmless for the correct onset of the radiation epoch (see Sec. V), the ultimate value that $\rho_\Lambda(H)$ takes at present, i.e., ρ_Λ^0, cannot be predicted within the model itself and hence can only be extracted from observations. Notice that if we could have the ability to predict this value it would be tantamount to solve the CC problem [13]. This is of course the toughest part of the longstanding unsolved cosmological constant problem. In our case, however, we have ascribed a new look to the problem, one that could perhaps make it more amenable for an eventual solution; namely, we have shown that the cosmological term which we have measured at present is not the same immutable tiny quantity that the ΛCDM assumes for the entire cosmic history, but rather a time-evolving variable that underwent a dramatic dynamical reduction from the inflationary time until the present days.

VIII. CONCLUSIONS

In this article we have proposed a new phenomenological scenario which provides a complete cosmic expanding history of the universe. It is based on a dynamical model (in fact an entire class of models) for the vacuum energy that covers all the relevant states of the cosmic evolution. The function $\Lambda = \Lambda(H)$ that we propose involves a power series of the Hubble rate H, which in practice consists of an additive term, a power H^2 and finally a higher power H^k ($k > 2$) which is responsible for the transition from the inflationary stage to the FLRW radiation epoch. The ansatz that we used is motivated by the covariance of the effective action of QFT in curved space-time and in this sense the even powers of H are preferred, although for completeness we have described the general case.

First of all the model itself predicts that the universe starts from a nonsingular state and thus we can solve easily the horizon problem. This early accelerated regime associated with the inflation has a natural ending by virtue of the faster decrease of the vacuum energy density thereby generating the radiation fluid and the ultrarelativistic gas particles. The novelty in the current work is the fact that the dynamical vacuum model that we propose smoothly accommodates the standard cosmic epochs characteristic of the ΛCDM model, namely the radiation-dominated, matter-dominated and late-time de Sitter phase ($\Lambda =$ const). The universe described in our proposal therefore evolves from a primeval de Sitter epoch to another late-time de Sitter epoch, which is the one we have recently entered. Let us note that the mechanism for inflation in our case is quite different from that of usual inflaton models. In this sense it may provide an alternative to them, especially after realizing that the PLANCK results [11] rule out some of these scalar field models, whereas in our case the sustained plateau we have in the vacuum inflationary phase could perhaps help explain better the new data and in particular the so-called "unlikeliness problem" [70]. A devoted analysis is of course needed, but it is clear that we remain as motivated as ever to look for new ideas and alternative mechanisms for inflation. Let us finally note that our model, apart from avoiding the initial singularity and alleviating the horizon and graceful exit problems, it also helps to mitigate the cosmological constant problem i.e., the fact that the observed value of the vacuum energy density ($\rho_\Lambda = c^2\Lambda/8\pi G \simeq 10^{-47}$ GeV^4) is many orders of magnitude below the value found using quantum field theory.

ACKNOWLEDGMENTS

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APPENDIX: GENERAL SOLUTIONS

1. From the early de Sitter stage to the ω-dominated phase

At early stages of the universe, the c_0 parameter is negligible and the Eq. (16) for the evolution of the Hubble function becomes

$$\dot{H} + \frac{3}{2}(1+\omega)H^2\left[1 - \nu - \alpha\left(\frac{H}{H_I}\right)^n\right] = 0. \tag{A1}$$

The integration of the above equation gives

$$H(a) = \frac{\tilde{H}_I}{[1 + Da^{n\xi}]^{1/n}}, \tag{A2}$$

where $\xi \equiv 3(1+\omega)(1-\nu)/2$ and $\tilde{H}_I \equiv H_I[(1-\nu)/\alpha]^{1/n}$. We stress that in our analysis we consider epochs of the cosmic evolution where matter is dominated by the relativistic or the nonrelativistic components, i.e., epochs where we have $\omega = 1/3$ and $\omega = 0$, respectively, without considering the interpolation regime between the two. Therefore, in practice for all the considerations in this section, we have $\omega = 1/3$—and so $\xi = 2(1-\nu)$—as our discussion is related to the transition from the initial de Sitter to the radiation-dominated universe. However, a simulation of the ω-dependence from $\omega = 0$ to $\omega = 1/3$ is done in Fig. 2.

In Eq. (A2), D is an integration constant that can be fixed using the condition $H(a_\star) \equiv H_\star$ [where $a_\star = a(t_\star)$, typically corresponding to the initial time $t_\star$ of the ω-fluid dominated era]. Thus,

$$D = a_\star^{-n\xi}\left[\left(\frac{\tilde{H}_I}{H_\star}\right)^n - 1\right], \tag{A3}$$

and it is greater than zero for $\tilde{H}_I > H_\star$. Note that if $D = 0$ the solution remains always de Sitter.

Using the auxiliary variable

$$u = -\frac{1}{Da^{n\xi}}, \tag{A4}$$

which transforms Eq. (A2) as

$$\dot{u} = -n\xi\tilde{H}_I u^{1+1/n}(u-1)^{-1/n}, \tag{A5}$$

and its inversion results:

$$\frac{dt}{du} = -\frac{1}{n\xi\tilde{H}_I}u^{-(1+1/n)}(u-1)^{1/n}. \tag{A6}$$

The second derivative may be put in the form:

$$u(1-u)\frac{d^2t}{du^2} + \left[1 + \frac{1}{n} - u\right]\frac{dt}{du} = 0. \tag{A7}$$

Hence, we have the hypergeometric equation with parameters $a = 0$, $b = 1/n$, and $c = 1 + 1/n$. Its integration yields

$$t(u) = B - Anu^{-1/n}F\left[-\frac{1}{n}, -\frac{1}{n}, 1 - \frac{1}{n}, u\right], \tag{A8}$$

where B and A are integration constants. We can set $B = 0$ if the origin of time is placed just after the inflation period and t is then the cosmic time in the FLRW epoch. Using Euler's relation for the hypergeometric function and the boundary condition (when $t = t_\star$ at the end of the inflationary period) for the Hubble parameter H the above solutions can be rewritten as

$$t(a) = B + \frac{(1 + Da^{n\xi})^{\frac{1+n}{n}}}{\xi\tilde{H}_I Da^{n\xi}}F\left[1, 1, 1 - \frac{1}{n}, \frac{-1}{Da^{n\xi}}\right], \tag{A9}$$

and for $n = 2$ this solution becomes

$$t(a) = B + \frac{1}{\xi H_I}\sqrt{\frac{\alpha(1 + Da^{2\xi})}{1-\nu}} - \frac{1}{\xi H_I} \times \sqrt{\frac{\alpha}{1-\nu}}\mathrm{ArcCoth}\sqrt{1 + Da^{2\xi}}. \tag{A10}$$

Using the Einstein equations and the above solutions we can obtain the corresponding energy densities:

$$\rho_\Lambda(a) = \tilde{\rho}_I\frac{1 + \nu Da^{n\xi}}{[1 + Da^{n\xi}]^{1+2/n}}, \tag{A11}$$

$$\rho(a) = \tilde{\rho}_I\frac{(1-\nu)Da^{n\xi}}{[1 + Da^{n\xi}]^{1+2/n}}, \tag{A12}$$

$$\rho_\mathrm{T}(a) = \tilde{\rho}_I\frac{1}{[1 + Da^{n\xi}]^{2/n}}, \tag{A13}$$

with $\tilde{\rho}_I \equiv 3\tilde{H}_I^2/8\pi G$. It is easy to check that these expressions correctly reproduce the energy densities we have used in Sec. V for the primeval de Sitter and radiation-dominated epochs.

2. From the ω-dominated era to the residual vacuum stage

Next we consider the derivation of the corresponding formulas for the more recent universe when the ω-fluid plus a vacuum fluid [described by (15)] expand under he condition $H \ll H_I$. In this case the evolution equation for the Hubble parameter Eq. (16) can be approximated as

$$aHH' + \xi H^2 - \frac{(1+\omega)}{2}c_0 = 0, \tag{A14}$$

where the prime denotes derivative with respect to the scale factor a, and again $\xi \equiv 3(1+\omega)(1-\nu)/2$. The first integral of this equation gives

$$H^2 = \frac{c_0}{3(1-\nu)}\left[\left(\frac{C_1}{a}\right)^{2\xi} + 1\right], \tag{A15}$$

where the constant

$$C_1{}^{2\xi} = a_0{}^{2\xi}\left[\frac{3H_0^2(1-\nu)}{c_0} - 1\right], \tag{A16}$$

is obtained from the condition $H(a_0) \equiv H_0$ today.

Using the above solutions, the Friedmann equations provide the total and the ω-fluid densities

$$8\pi G\rho_T(a) = \frac{c_0}{1-\nu}\left[\left(\frac{C_1}{a}\right)^{2\xi} + 1\right], \tag{A17}$$

$$8\pi G\rho(a) = c_0\left(\frac{C_1}{a}\right)^{2\xi}. \tag{A18}$$

In a more explicit form, the Hubble function (A15) reads

$$H^2(a) = \frac{H_0^2}{1-\nu}[\Omega_X^0 a^{-2\xi} + \Omega_\Lambda^0 - \nu], \tag{A19}$$

where we have the sum rule $\Omega_X^0 + \Omega_\Lambda^0 = 1$, and we have set $\omega = 0$ $(X = m)$ since we are in the matter-dominated epoch. The ω-fluid density (A18) can be expressed as

$$\rho(a) = \rho^0 a^{-2\xi}, \tag{A20}$$

where ρ^0 is the current value. We can see that for $\nu = 0$ we retrieve the standard scaling $\rho = \rho^0 a^{-3(1+\omega)}$. The departure from this law caused by a nonvanishing ν is related to the exchange of energy between matter and vacuum. By the same token the vacuum is no longer static, and the effective CC evolves as

$$\Lambda(a) = \frac{c_0}{1-\nu}\left[\nu\left(\frac{C_1}{a}\right)^{2\xi} + 1\right]. \tag{A21}$$

The corresponding vacuum energy density is the following:

$$\rho_\Lambda(a) = \rho_\Lambda^0 + \frac{\nu\rho^0}{1-\nu}[a^{-2\xi} - 1]. \tag{A22}$$

We see that only for $\nu = 0$ we recover $\Lambda = c_0 =$ const and $\rho_\Lambda(a) = \rho_\Lambda^0 =$ const, as in the ΛCDM case. Furthermore, we can easily check that Eqs. (A20) and (A22) satisfy the overall local conservation law (8), which can be rewritten in terms of the scale factor as follows:

$$\rho'_\Lambda(a) + \rho'(a) + \frac{3}{a}(1+\omega)\rho(a) = 0, \tag{A23}$$

where the prime indicates differentiation with respect to the scale factor.

We can integrate Eq. (A15) to obtain the time evolution of the scale factor $a(t)$:

$$a(t) = C_1 \sinh^{1/\xi}\left[\sqrt{3c_0(1-\nu)}(1+\omega)(t-C_2)/2\right]. \tag{A24}$$

Without losing the generality we can set $C_2 = 0$. Substituting (A24) in the previous equations we immediately get the time-evolving functions $\rho = \rho(t)$ and $\Lambda = \Lambda(t)$.

Let us finally mention for completeness that there are cases where we have to deal with a mixture of cold matter and radiation. Defining Ω_m^0 and Ω_r^0 as the standard nonrelativistic and radiation density parameters at the present time, one can show that the complete Hubble function reads

$$H^2(a) = \frac{H_0^2}{1-\nu}[\Omega_m^0 a^{-3(1-\nu)} + \Omega_\Lambda^0 + \Omega_r^0 a^{-4(1-\nu)} - \nu], \tag{A25}$$

where the density parameters satisfy the extended sum rule $\Omega_m^0 + \Omega_r^0 + \Omega_\Lambda^0 = 1$.

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